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Gas kinetic forces on thin plates in the presence of thermal gradients

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Gases perturbed by thermal gradients exert forces on surfaces exposed to the molecular bombardment. We calculate the force on thin plates with temperature differences between the opposite faces in dense gases. Using the Chapman-Enskog method we obtain an analytical formula that accounts for normal and shear stresses on the plate edge. We predict a force per unit length of the plate perimeter up to 6.6×10^{-8} N/mm per degree Kelvin of temperature difference in air. The present result is compared with experimental data from the literature finding an excellent agreement in those cases where experimental conditions and procedures permit comparison. The proposed analytical formula provides an effective tool for calculating forces in microstructures and nanostructures affected by temperature inequalities.

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I. INTRODUCTION

An interesting class of gas kinetic phenomena occurs in gases with thermal gradients. Notable examples are thermal creeping along unequally heated walls [1,2], thermal transpiration across porous membranes [3,4], and steady state pressure gradients along tubes in Knudsen-type pumps [5,6]. Thermal gradients also give rise to macroscopic forces at the gas-surface interface such as thermophoretic [7,8] and radiometric forces [9–11]. These play a role in the dynamics of particles and aerosols in the earth atmosphere and are of increasing concern in microelectromechanical systems (MEMS) [12–14] where temperature inequalities on the microscale may give rise to unwanted stresses. The forces are governed by the Knudsen number Kn which is the ratio of the molecular mean free path λ to a characteristic length scale τ of the object immersed in the gas. Typically the forces increase as the gas pressure is decreased, they reach a maximum for a certain Kn range, and then decrease for vanishing gas densities. For this reason thermal forces are traditionally associated to rarefied gas science. However, advances in microengineering and nanoengineering allow to achieve large Knudsen numbers and thus larger forces at higher pressures due to the characteristically smaller dimensions of microstructures and nanostructures.

This paper investigates forces exerted by dense gases on thin plates whose opposite faces are kept at different temperatures. Dense gases are characterized by the dominance of intermolecular collisions over molecule-surface collisions. Within this regime we focus here on plates whose side a is larger than the mean free path λ, but whose thickness τ is comparable to λ. These objects, as a sort of link between the microscopic and the macroscopic domains, are common in semiconductor science and in microdevices and nanodevices which are built around thin films deposited through chemical or physical vapor deposition. Thin plates are used as microcantilevers in atomic force microscopy [15,16] and in high-precision Casimir force measurements [17]. These devices are frequently heated on one side by a laser for measuring the cantilever displacement; the laser generates a thermal gradient across the plate and in the surrounding gas, and the associated thermal forces can affect the measurements. In other cases the thermal forces are exploited for powering MEMS actuators [12,14]. The rapidly growing discipline of thin-film thermoelectrics [18,19] offers the opportunity to test thermal forces in Peltier microcoolers and nanocoolers. These devices can generate temperature differences as high as 15 K across a 700 nm thick plate at atmospheric pressure [20] corresponding to thermal gradients in the 10^7 K/m range. These are among the largest gradients ever observed in nature or in a laboratory and are expected to generate significant gas kinetic forces when the Peltier devices are detached from the substrate.

The present investigation also seeks basic understanding of momentum transport phenomena at high gas densities. It is known that for $a \ll \lambda$ (free-molecular regime) thermal forces are proportional to the plate area [21]; however for $a \gg \lambda$ the contribution from the edges is predominant [22]. Dependence of the force on the plate perimeter is consistent with experiments performed in Refs. [23,24] and with recent measurements by Passian *et al.* [13] who tested plates with etched microholes. A plate with a hot and a cool surface experiences normal and shear stresses due to molecules impinging on the edge. This is depicted in Fig. 1. Shear forces were first investigated in Refs. [1,3] and lead to gas streaming along surfaces parallel to the thermal gradient. The stresses associated with the streaming have been studied using a hydrodynamic approach in Refs. [25,26]. Normal forces are due to a local excess pressure present on the area

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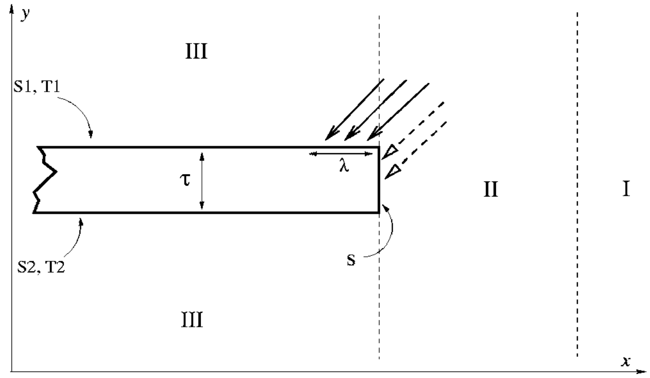

FIG. 1. A large plate with one edge immersed in a gas. The two opposite faces have different temperatures, generating a thermal gradient in region II. The solid arrows represent molecules contributing to the normal force. The dashed arrows contribute to the shear force.

of a strip a mean free path wide along the perimeter, as elucidated by Einstein in Ref. [27]. In his contribution to the study of radiometric forces, Einstein derived a formula for the force acting on an infinitely thin plate in a dense gas with a temperature gradient. Einstein finds a normal force proportional to the plate perimeter; however his work is a qualitative treatment which does not compute the exact constant factors; as such the agreement with the observations is modest [24]. In this paper we present a more comprehensive and rigorous calculation of the thermal forces acting on the edges of flat plates immersed in dense gases. We calculate both the shear and the normal contributions using the Chapman-Enskog distribution of velocities. While today's numerical methods allow to find exact solutions to the Boltzman equation, those methods frequently involve lengthy computations and in some cases require computers with dedicated software. The approach proposed here offers an approximated, yet rapid and analytical solution. As the results given below show, our approach provides—in the example given—a sufficently adequate platform for calculating thermal forces.

II. THE MODEL

Consider a plate with two facial surfaces $S1$ and $S2$ and an edge surface s as shown in Fig. 1. The plate has a side length a, a perimeter ℓ, and a thickness $\tau \ll a$. The plate is immersed in a gas consisting of molecules of mass m with number density n. The present model ignores all degrees of freedom other than translatory motion. The gas and the plate are located in a space with orthogonal coordinates x, y, z and the plate is oriented parallel to the x-z plane. The surfaces $S1$ and $S2$ are kept at different temperatures T_1 and T_2, respectively, with $T_1 > T_2$ and $\Delta T = T_1 - T_2$. We will divide the gas in three regions as shown in Fig. 1. Region I is the unperturbed gas far away from the plate. In this region the temperature is T_0 and a Maxwellian distribution of velocities holds. Region III defines the column of gas above $S1$ and below $S2$. In this region a mild temperature gradient is present; for instance above $S1$ the temperature falls from T_1 to T_0 on a length scale a along the y axis. Since a is large, this gradient will be ignored. In region II a strong temperature gradient given by $dT/dy = \Delta T/\tau$ is present, this is the relevant gradient in the problem. Any temperature gradients along the x-z plane are neglected. Region II stretches for several mean free paths from the plate edge. In all three regions I, II, III, the pressure is assumed to be a constant, therefore the product nT is an invariant [28], furthermore we assume the gas to be in a steady state. In the presence of temperature gradients the gas is governed by a non-Maxwellian velocity distribution given by $f = f_0 + f_1$, where f_0 denotes the Maxwell-Boltzmann distribution function and f_1 is the Chapman-Enskog term [29] which is linear in the gradient dT/dy. Following Ref. [30], the complete distribution function of velocities reads

$$f(\vec{v}) = \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta(v_x^2+v_y^2+v_z^2)}\left[1 + A v_y\left(\frac{5}{2} - \beta(v_x^2+v_y^2+v_z^2)\right)\right], \tag{1}$$

where

$$\beta = \frac{m}{2kT},$$

and the gradient term A is obtained by solving the scattering problem for molecules with cross section $\pi\sigma^2$,

$$A = \frac{15}{32 n \pi \sigma^2 T}\sqrt{\frac{\pi m}{kT}}\frac{dT}{dy},$$

where k is the Boltzmann constant. Details of the derivation of (1) can be found in either Ref. [29] or Ref. [30]. The Chapman-Enskog approximation (1) is valid for small to moderate temperature gradients. For large gradients higher orders in the expansion are required. Furthermore Eq. (1) is inadequate for very large Knudsen numbers where intermolecular collisions become irrelevant (see discussion below).

III. CALCULATION OF THE NORMAL FORCE

The molecules from region II that impinge obliquely on $S1$ (solid arrows in Fig. 1) impart a larger momentum in the y direction than molecules impinging on $S2$ as the former ones have a higher mean velocity. Thus a minute portion of $S1$ near the plate border suffers a higher pressure through molecular bombardment. The momentum p_1 per unit area imparted to $S1$ by molecules impinging with negative v_x and negative v_y is given by

$$p_1 = nm\int_{-\infty}^{0} dv_x \int_{-\infty}^{\infty} dv_z \int_{-\infty}^{0} dv_y v_y^2 f(\vec{v}) = \frac{nkT}{4} + \frac{15k}{64\sqrt{2}\pi\sigma^2}\frac{\Delta T}{\tau}. \tag{2}$$

The right-hand side of Eq. (2) shows a Maxwellian term and a gradient term. The momentum imparted by molecules impinging on $S2$ is

$$p_2 = nm\int_{-\infty}^{0} dv_x \int_{-\infty}^{\infty} dv_z \int_{0}^{\infty} dv_y v_y^2 f(\vec{v}) = \frac{nkT}{4} - \frac{15k}{64\sqrt{2}\pi\sigma^2}\frac{\Delta T}{\tau}. \tag{3}$$

In order for Eq. (3) to be physical its right-hand side must be positive,

$$\frac{nkT}{4} \geqslant \frac{15k}{64\sqrt{2}\pi\sigma^2}\frac{\Delta T}{\tau}, \tag{4}$$

this imposes an upper limit on the magnitude of the gradient,

$$\frac{15}{16}\lambda\frac{dT}{dy} \leqslant T \tag{5}$$

and on the Knudsen number

$$\frac{15}{16}\mathrm{Kn} \leqslant \frac{T}{\Delta T}. \tag{6}$$

When either the gradient or the Knudsen number exceed the above limits the Chapman-Enskog approximation proposed in this paper is not applicable. The difference between (2) and (3) gives the net normal momentum per unit area. The resulting normal force is

$$\Delta f_{\mathrm{norm}} = (p_1 - p_2)(\lambda\ell) = \frac{15k}{32\sqrt{2}\pi\sigma^2}\frac{\Delta T}{\tau}\lambda\ell.$$

Here the Maxwellian terms have cancelled leaving the sum of the gradient terms. This is consistent with a force vanishing for homogeneous temperature distributions. In deriving Δf_{norm} it is assumed, following Ref. [27], that the area involved in the molecular bombardment is a strip a mean free path wide along the plate perimeter ℓ. This assumption requires that the mean free path is smaller than the side of the plate,

$$0 < \lambda \leqslant a,$$

which imposes a further constraint on the Knudsen number. We now calculate the contribution from the molecules reflected off the plate. Molecules that thermalize with the surfaces $S1$ and $S2$ will have a Maxwellian distribution of velocities. Thus the net recoil momentum from thermalized molecules is of the form

$$kn_1 T_1 - kn_2 T_2,$$

where n_1 and n_2 refer to the gas densities in the layers immediately above the hot and cold surfaces, respectively. Because the product $n(y)T(y)$ is constant throughout the gas, this difference vanishes. Thus the contribution from accommodated molecules is zero. Conversely molecules that undergo an elastic reflection maintain their initial energy and velocity distribution. Their contribution to the force equals in sign and modulus that of the impinging molecules [31]. Based on this, the complete normal force can be approximated by

$$\Delta f_{\mathrm{norm}}^{\alpha} = (2-\alpha)\frac{15}{32\sqrt{2}\pi}\frac{k}{\sigma^2}\frac{\Delta T}{\tau}(\lambda\ell), \tag{7}$$

where α is the energy accommodation coefficient of the plate assumed to be the same on both $S1$ and $S2$. Equation (7) is valid for $\tau > \lambda$. For plates with thickness $\tau \leqslant \lambda$ the thermal gradient in region II becomes

$$(dT/dy)_{<} = \Delta T/\lambda$$

which is the maximum [32] effective gradient for a given ΔT. Accordingly the normal force on thin plates reads

$$F_{\mathrm{norm}} = (2-\alpha)\frac{15}{32\sqrt{2}\pi}\frac{k}{\sigma^2}\Delta T\ell. \tag{8}$$

Equation (8) is independent of the gas density as long as $\tau \leqslant \lambda$.

IV. CALCULATION OF THE SHEAR FORCE

In addition to the normal force the plate suffers a shear stress due to molecules impinging on the surface s (dashed arrows in Fig. 1). Again, the molecules coming from the hot side carry a higher momentum in the y direction than molecules coming from the cold side. Accordingly the area s undergoes a net shear pressure directed toward the cold side given by

$$\Delta p_{\mathrm{shear}} = nm\int_{-\infty}^{0} dv_x \int_{-\infty}^{\infty} dv_z \int_{-\infty}^{\infty} dv_y v_y v_y v_x f(\vec{v}) = \frac{15}{64\sqrt{2}\pi}\frac{k}{\sigma^2}\frac{\Delta T}{\tau}. \tag{9}$$

Furthermore, using Eq. (1) we find that the contribution from molecules leaving the surface s is zero in the case of completely accommodated molecules. For molecules undergoing a specular reflection the contribution is equal and opposite in sign to (9). Thus specular reflections reduce the shear effect. The total accommodated shear pressure reads

$$\Delta p_{\mathrm{shear}}^{\alpha} = \frac{15}{64\sqrt{2}\pi}\frac{k}{\sigma^2}\frac{\Delta T}{\tau}\alpha.$$

The shear force for $\tau \leqslant \lambda$ is

$$F_{\mathrm{shear}} = \frac{15}{64\sqrt{2}\pi}\frac{k}{\sigma^2}\alpha\frac{\Delta T}{\lambda}(\tau\ell). \tag{10}$$

V. COMPLETE GAS-KINETIC FORCE AND COMPARISON WITH EXPERIMENTS

The complete gas-kinetic force acting on the plate edge is given by the sum of the normal (8) and shear (10) contributions,

$$F = \phi\Delta T\ell\left[2 + \alpha\left(\frac{1}{2}\frac{1}{Kn} - 1\right)\right], \tag{11}$$

where

$$\phi = \frac{15}{32\sqrt{2}\pi}\frac{k}{\sigma^2}, \quad Kn = \frac{\lambda}{\tau}.$$

Formula (11) is the main result of the present paper. According to (11) the gas-kinetic force on a thin plate is linearly dependent on the perimeter length and on the temperature difference ΔT. Because of the limits on the temperature gradient and on the Knudsen number mentioned above, Eq. (11) is only valid for

$$1 \leqslant Kn \leqslant \frac{a}{\tau}, \quad 1 \leqslant Kn \leqslant \frac{16}{15}\frac{T}{\Delta T}. \tag{12}$$

Equation (11) has a maximum for $\tau = \lambda$ and $\alpha = 0$. If the ambient gas is air at 15 °C, and the gas molecules are modeled as hard spheres of diameter $\sigma = 0.372$ nm, as proposed in Ref. [29], we find that the maximum force per unit length of plate perimeter is 6.6×10^{-8} N/mm per degree Kelvin of temperature difference. The dependence on α is weak, the force varying by less than 25% in the $0 < \alpha < 1$ interval. To check the validity of our result we have compared Eq. (11) against experimental data from the literature. In Ref. [24] Marsh and collaborators measured the thermal force on several thin mica plates having different perimeters but the same facial area. The investigation was performed in air in a pressure range for which the mean free path was smaller than the dimensions of the vessel and of the plates. The mica plates had one face blackened with soot and this face was heated by means of a focused light beam. The temperatures on the opposing faces were monitored with thermocouples and the gas-kinetic force was measured with a torsion pendulum. Forces on plates with larger perimeter were found to be larger. The force measured on plates with thickness $\tau = 450$ μm, at 0.03 ± 0.001 Torr pressure ($\lambda = 1.62$ mm) and at $\Delta T = 0.49 \pm 0.05$ °C, was found to be 0.02 dyne per each cm of plate perimeter. The accommodation coefficients are not given in Ref. [24]. However for unpolished mica and carbon soot, values between 0.7 and 1 are plausible. References [33] and [34] provide values of $\alpha = 0.80$ for mica and $\alpha = 0.909$ for soot, respectively. We take the mean of these values and insert $\alpha = 0.85 \pm 0.05$ in formula (11). The resulting calculated force is shown in Table I along with the value predicted by Ref. [27]. The uncertainty in the calculated force is ±0.004 dyne/cm. The mismatch between Eq. (11) and the observation is smaller than the uncertainty.

TABLE I. Comparison of calculated and observed force per unit length of plate perimeter for a 0.45 mm thick plate. The plate has a temperature difference $\Delta T = 0.49$ °C between its opposing faces, and is immersed in air at 0.03 Torr.

	Ref. [27]	Eq. (11)	Observed
Force (dyne/cm):	0.03	0.0205	0.02

VI. CONCLUSIONS

In summary we have found an analytical formula for calculating gas-kinetic forces on thin, unequally heated plates. This is given by Eq. (11) which expresses the force in terms of the perimeter length ℓ. We have found good agreement between Eq. (11) and observations, although further testing will be needed to confirm this finding. A comparison with more recent observations in the transitional Knudsen regime such as those performed by Lereu *et al.* [35] could shed more light. The authors intend to perform the comparison in a future paper. Our investigation strengthens the picture of thermal forces occurring at the plate edge in the regime considered. As long as the plate is thin, the mass motion of the gas can be neglected and the force can be successfully computed using the gas-kinetic approach proposed here. Equation (11) has been derived for a square plate, however it is expected to hold for plates of different shapes as long as $\ell \gg \tau$. Of particular interest are plates with enhanced perimeters through etched holes and slits [13], and possibly microdiffraction gratings. Future work should be directed toward investigating gases with additional degrees of freedom besides transitory motion. Finally it would be interesting to implement a more rigorous model for the accommodation coefficients and consider facial surfaces $S1$ and $S2$ made of two different materials as this is frequently the case in experimental situations.

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